Exercise Sheet 5: Pointed and Unpointed Homotopy Sets

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1 Pointed and Unpointed Homotopy Sets

Please complete all the exercises, which this week have been ripped shamelessly from pg. 139 of Jeff Strom's book [1]. Please note, however, that our notation differs from the source material.

Recall the functor

$$Top_* \xrightarrow{U} Top \tag{1.1}$$

that forgets basepoints. Since a pointed homotopy also defines an unpointed homotopy this functor is *homotopical* in the sense introduced in Exercise Sheet 1. This means that the composite of U with $\gamma : Top \to hTop$ is a homotopy functor, and in particular induces a unique functor

$$hTop_* \xrightarrow{U} hTop. \tag{1.2}$$

This gives rise to a function

$$[X, Y] \to [X, Y]_0 \tag{1.3}$$

allowing for a comparison of the pointed and unpointed homotopy sets. In general these sets may be very different. It is the purpose of the current exercises to try to understand this difference.

Recall the functor

$$Top \xrightarrow{(-)_+} Top_*, \qquad X \mapsto X_+ \tag{1.4}$$

which adds a disjoint basepoint and is left adjoint to U.

Exercise 1.1 Let X be an unpointed space. Show that the pointed space X_+ is well-pointed. $\square$

Exercise 1.2 Let $j : A \hookrightarrow X$ be a cofibration in Top. Show that $j_+ : A_+ \to X_+$ is a pointed cofibration between well-pointed spaces in Top_* . $\square$

This means that there is a strict cofiber sequence

$$A_+ \xrightarrow{j_+} X_+ \xrightarrow{q} X/A \tag{1.5}$$

in the sense defined in Exercise Sheet 4. Thus given a pointed space Y we can apply $[-, Y]$ and make sense of an exact sequence of pointed homotopy sets. At the moment, however, we'll only be interested in the simplest possible case.

Exercise 1.3 Let X be a based space. Then the statement that $* \hookrightarrow X$ is a cofibration in Top is the statement that X is almost well-pointed. Assuming this show that there is a strict cofiber sequence

$$*_+ \to X_+ \to X \tag{1.6}$$

and use this to derive the existence of a short exact sequence of pointed sets

$$0 \leftarrow \pi_0 Y \leftarrow [X, Y]_0 \leftarrow [X, Y] \leftarrow 0 \tag{1.7}$$

when Y is a given based space. (Hint: show that $*_+ \to X_+$ has a retraction $X_+ \to *_+$.) $\square$

Exercise 1.4 Assume that X, Y are based spaces with X is well-pointed. What is a sufficient condition to conclude that $[X, Y] \to [X, Y]_0$ is surjective? $\square$

Next we'll want a way to understand the other end of (1.7). We'll keep working with a well-pointed space X and an arbitrary based space Y. Assume given pointed maps $f, g : X \to Y$ and a *free* homotopy $H_t : f \simeq g$. Then $H_0(*) = f(*) = *$ and $H_1(*) = g(*) = *$, but H_t need need not preserve basepoints. We'll say that the **loop** of H is the map

$$\lambda_H : S^1 \to Y, \qquad t \mapsto H_t(*). \tag{1.8}$$

Of course here we are viewing S^1 as the quotient $I/\partial I$. Note that λ_H is a based map.

Exercise 1.5 Let X be a well-pointed space and let G be a path-connected topological group.

1. Show that for any pointed maps $f : X \to G$ and $\alpha : S^1 \to G$, there is a free homotopy $F : f \simeq f$ with loop $\lambda_F = \alpha$.

2. Show that if $H : f \simeq g$ is a free homotopy of pointed maps $f, g : X \to G$, then there is a free homotopy $K : f \simeq g$ whose loop is null homotopic.

3. Show that the comparison map $[X, G] \to [X, G]_0$ is bijective.

4. Conclude that $\pi_1 S^1 \to [S^1, S^1]_0$ is bijective. $\square$

If we replace the topological group G by an arbitrary pointed space Y, then even if we assume Y to be path-connected, the comparison map $[X, Y] \to [X, Y]_0$ may not be injective. We have some insight from the last exercise into what can go wrong. It turns out that the homotopy group $\pi_1 Y$ operates on $[X, Y]$, with $[X, Y]_0$ as the quotient. We'll return to this at a later point.

References

[1] J. Strom, *Modern Classical Homotopy Theory*, American Mathematical Society, (2011).